Pertanika J. Sci. & Technol. 31 (2): 843 - 873 (2023)

SCIENCE & TECHNOLOGY

Journal homepage: http://www.pertanika.upm.edu.my/

Efficient Frequency-Dependent Coefficients of Explicit Improved Two-Derivative Runge-Kutta Type Methods for Solving Third-Order IVPs

Lee Khai Chien[1]*, Norazak Senu[1,2], Ali Ahmadian[3] and Siti Nur Iqmal Ibrahim[1,2]

[1]*Institute for Mathematical Research, Universiti Putra Malaysia, 43400 UPM, Serdang, Selangor, Malaysia*
[2]*Department of Statistics and Mathematics, Universiti Putra Malaysia, 43400 UPM, Serdang, Selangor, Malaysia*
[3]*Institute of Industry Revolution 4.0, The National University of Malaysia, 43600 UKM, Bangi, Selangor, Malaysia*

ABSTRACT

This study aims to propose sixth-order two-derivative improved Runge-Kutta type methods adopted with exponentially-fitting and trigonometrically-fitting techniques for integrating a special type of third-order ordinary differential equation in the form $u'''(t) = f(t, u(t), u'(t))$. The procedure of constructing order conditions comprised of a few previous steps, k_{-i} for third-order two-derivative Runge-Kutta-type methods, has been outlined. These methods are developed through the idea of integrating initial value problems exactly with a numerical solution in the form of linear composition of the set functions $e^{\omega t}$ and $e^{-\omega t}$ for exponentially fitted and $e^{i\omega t}$ and $e^{-i\omega t}$ for trigonometrically-fitted with $\omega \in \mathbb{R}$. Parameters of two-derivative Runge-Kutta type method are adapted into principle frequency of exponential and oscillatory problems to construct the proposed methods. Error analysis of proposed methods is analysed, and the computational efficiency of proposed methods is demonstrated in numerical experiments compared to other existing numerical methods for integrating third-order ordinary differential equations with an exponential and periodic solution.

Keywords: Exponentially-fitted, third-order ordinary differential equations, trigonometrically-fitted, two-derivative runge-kutta methods

ARTICLE INFO

Article history:
Received: 21 May 2022
Accepted: 05 October 2022
Published: 06 March 2023

DOI: https://doi.org/10.47836/pjst.31.2.10

E-mail addresess:
kclee_1017@hotmail.com (Lee Khai Chien)
norazak@upm.edu.my (Norazak Senu)
ali.ahmadian@ukm.edu.my (Ali Ahmadian)
iqmal@upm.edu.my (Siti Nur Iqmal Ibrahim)
*Corresponding author

ISSN: 0128-7680
e-ISSN: 2231-8526

INTRODUCTION

Third-order ordinary differential equations (ODEs) are widely applied in the fields of engineering and physic, such as thermohaline convection, incompressible Newtonian fluid, turbulent transport of viscoelastic fluids in penetrable channels, jerk mechanical system with jerk curves, thin film flow systems and other disciplines (Herrera, 2019; Allogmany & Ismail, 2020).

This article focuses on developing an improved two-derivative Runge-Kutta type method with exponentially and trigonometrically fitting techniques based on frequency evaluation for integrating third-order ODEs with exponential or oscillatory solution (Equation 1).

$$\begin{aligned} &u'''(t) = f(t, u(t), u'(t)) \\ &u(t_0) = u_0, \qquad u'(t_0) = u'_0, \qquad u''(t_0) = u''_0. \end{aligned} \tag{1}$$

In recent, many research are widely studied by researchers regarding the characteristics of solutions with frequency-dependent properties and the development of efficient methods to solve ODEs with exponential and oscillatory solutions to illustrate the model of application problems such as orbital mechanics, molecular dynamics and electronics, Van der Pol's equations, Kepler's problem in a dynamical system, Bessel equations and harmonic oscillator (Franco & Randez, 2018; Ahmad et al., 2020).

Simos and Williams (1999) constructed exponentially and trigonometrically fitted Runge-Kutta methods with order three for solving the Schrödinger equation. The numerical results proved the efficiency of the proposed methods. Then, Zhang et al. (2013) extended Simos and Williams' works by proposing fifth-order trigonometrically fitted two-derivative Runge-Kutta (TDRK) methods for solving the Schrödinger equation. The stability and phase properties of the proposed methods are investigated. Some research concentrate on solving application problems through TDRK methods. Chen et al. (2015) applied newly developed TDRK method oscillatory properties in integrating oscillatory genetic regulatory systems, which is important to illustrate the chemical reaction in living cells. Monovasilis and Kalogiratou (2021) developed amplification-fitted and phase-fitted seventh-order TDRK methods with frequency-reliant coefficients. Proposed methods are built based on minimised dispersion and dissipation error, leading to high efficiency in solving Stiefel and Bettis and harmonic oscillator problems.

Also, some research is on constructing direct methods with exponential and trigonometric fitting techniques for integrating high-order ODEs with exponential and trigonometric solutions. D'Ambrosio et al. (2014) revised the multistage Runge-Kutta-Nyström method with the exponentially-fitting technique for integrating special second-order ODEs with periodic or oscillatory solutions. Demba et al. (2016) applied the Simos technique in developing the symplectic explicit Runge-Kutta-Nyström method with a trigonometrically-fitting technique. Mei et al. (2017) implemented finite-energy conditions into the traditional Runge-Kutta-Nyström method to solve nonlinear wave equations. Zhai et al. (2018) constructed implicit symplectic and symmetric and exponentially-fitted Runge-Kutta-Nyström with linear combinations of exponential functions to solve second-order oscillatory problems. Two years later, Samat and Ismail (2020) derived a four-stage explicit sixth-order hybrid method with variable steps based on a trigonometrically-fitting technique. Demba et al. (2020) developed 5(3) embedded explicit Runge-Kutta-Nyström methods with an exponentially-fitting technique to reduce the computational cost in error estimation for solving special second-order ODEs with a periodic solution. The variable step-size technique is utilised for the derivation of the methods, and the numerical results proved the efficiency of the methods by generating more accurate results than other existing methods.

Some researchers are interested in developing the Improved Runge-Kutta (IRK) method, which comprises a few previous terms in the formulation to compute the future value. Several previous terms, such as *b*-*i* and *k*-*i* are inserted in the formulation to improve the method's accuracy in numerical integration. Rabiei (2011) proposed the improved Runge-Kutta methods and attained an order of up to five. Later, the stability of the methods was discussed as well. Rabiei and Ismail (2012) also constructed the improved Runge-Kutta method for solving first and second ODEs by presenting the new terms *k*-*i*, which is the previous step of k_i. Another modified, improved Runge-Kutta fifth-order five-stage technique is proposed by Senthilkumar et al. (2013) to solve a second-order robot arm problem. The study illustrates the importance of improved methods of visualising application problems. IRK method is not only used to solve first-order ODEs, but some researchers utilised IRK method to solve high-order ODEs or other types of differential equations. Hussain et al. (2017) emphasised solving high-order ODEs, and they developed a fourth-order improved RK method with a lower number of function evaluations for solving third-order ODEs directly. The stability polynomial of the proposed method was studied, and the great numerical performance of the method was proved by yielding a low maximum absolute error. Tang and Xiao (2020) modified the classical IRK methods into improved Runge-Kutta-Chebyshev methods based on spatial discretisation of partial differential equations (PDEs). The width of the stability domain increased significantly, and the proposed methods are applied to solve several numerical problems, including advection-diffusion-reaction equations with dominating advection.

However, there is no improved two-derivative Runge-Kutta type method with exponentially and trigonometrically fitting techniques developed in the current research field for solving high-order ODEs. Thus, we propose a three-stage sixth-order explicit improved two-derivative Runge-Kutta type method with exponentially and trigonometrically fitting techniques, denoted as EFIRKT6 and TFIRKT6 methods, to solve third-order ODEs with exponential and oscillatory solutions. Nonoscillatory and oscillatory properties of third-order ODEs are discussed. We developed order conditions for an improved TDRKT method. Then, exponentially-fitted and trigonometrically-fitted improved TDRKT methods are derived by adapting a linear combination of frequency-dependent coefficients and constructed EFIRKT6 and TFIRKT6 methods. Analysis of error for EFIRKT6 and TFIRKT6 methods was discussed. The numerical performance of EFIRKT6 and TFIRKT6 methods and other existing numerical methods are shown. This article ends with a discussion and conclusion.

METHODOLOGY

The explicit three-stage sixth-order two-derivative improved Runge-Kutta type method with exponentially and trigonometrically fitting techniques are proposed. Here we introduce the criteria for achieving nonoscillatory and oscillatory properties of third-order ordinary differential equations.

Oscillatory and Nonoscillatory Standard for Third-Order Linear Differential Equations

The oscillatory and nonoscillatory standards for third-order ordinary differential equations are mentioned as follows (Lee et al., 2020):

$$u'''(t) + \alpha(t)u'(t) + \beta(t)u(t) = 0. \tag{2}$$

Equation 2 consists of an oscillatory solution if both α(t) and β(t) are constant, negative and fulfil the following requirement in Equation 3:

$$-\beta(t) - \frac{2}{3\sqrt{3}}\left(\alpha(t)\right)^{\frac{3}{2}} > 0. \tag{3}$$

then two linear independent oscillatory solutions exist, and zeroes of any oscillatory solutions are split in which the oscillatory solution of Equation 2 is a linear combination of them (Lazer, 1966). The solution of Equation 2 is oscillatory iff it contains an infinite number of zeroes in (0, +∞) and nonoscillatory iff it contains a finite number of zeroes in (0, +∞). We focus on β(t) = 0 as follow:

1. $u'''(t) = \alpha(t)u'(t),\ \alpha(t) > 0,$ the solution of characteristic roots equations contains exponential function if those equations contain two real solutions and one zero.
2. $u'''(t) = -\alpha(t)u'(t),\ \alpha(t) > 0,$ the solution of characteristic roots equations contains an oscillatory function if those equations contain one real solution and two conjugate roots.

Exponentially-Fitted and Trigonometrically-Fitted Two-Derivative Improved Runge-Kutta Type Methods

The criteria for developing an explicit two-derivative improved Runge-Kutta type method with exponentially and trigonometrically fitting techniques denoted as EFIRKT and TFIRKT methods will be proposed. Here we include two parameters, γ_i and $\hat{\gamma}_i$, into U_i' and U_i'' to implement oscillatory properties into the formulation of the two-derivative Runge-Kutta-type TDIRKT method as in Equation 4.

$$u_{n+1} = u_n + \frac{3}{2}hu_n' - \frac{1}{2}hu_{n-1}' + \frac{5h^2}{12}(u_n'' - u_{n-1}'')$$
$$+\frac{h^3}{6}[f(t_n, u_n(t), u_n'(t)) - f(t_{n-1}, u_{n-1}(t), u_{n-1}'(t))]$$
$$+h^4 \sum_{i=2}^{s} b_i\left[g\left(t_n + c_i h, U_i(t), U_i'(t), U_i''(t)\right) - g\left(t_{n-1} + c_i h, U_{-i}(t), U_{-i}'(t), U_{-i}''(t)\right)\right]$$
$$u_{n+1}' = u_n' + \frac{3}{2}hu_n'' - \frac{1}{2}hu_{n-1}'' + \frac{5h^2}{12}[f(t_n, u_n(t), u_n'(t)) - f(t_{n-1}, u_{n-1}(t), u_{n-1}'(t))]$$
$$+h^3 \sum_{i=2}^{s} b_i'\left[g\left(t_n + c_i h, U_i(t), U_i'(t), U_i''(t)\right) - g\left(t_{n-1} + c_i h, U_{-i}(t), U_{-i}'(t), U_{-i}''(t)\right)\right]$$
$$u_{n+1}'' = u_n'' + \frac{3}{2}hf(t_n, u_n(t), u_n'(t)) - \frac{1}{2}hf(t_{n-1}, u_{n-1}(t), u_{n-1}'(t))$$
$$+h^2 \sum_{i=2}^{s} b_i''\left[g\left(t_n + c_i h, U_i(t), U_i'(t), U_i''(t)\right) - g\left(t_{n-1} + c_i h, U_{-i}(t), U_{-i}'(t), U_{-i}''(t)\right)\right]$$

where

$$U_i = u_n + c_i h u_n' + \frac{(c_i h)^2}{2}u_n'' + \frac{(c_i h)^3}{6}f(t_n, u_n, u_n')$$
$$+h^4 \sum_{j=1}^{s} A_{i,j} g\left(t_n + c_i h, U_j(t), U_j'(t), U_j''(t)\right)$$
$$U_{-i} = u_{n-1} + c_i h u_{n-1}' + \frac{(c_i h)^2}{2}u_{n-1}'' + \frac{(c_i h)^3}{6}f(t_{n-1}, u_{n-1}, u_{n-1}')$$
$$+h^4 \sum_{j=1}^{s} A_{i,j} g\left(t_{n-1} + c_i h, U_{-j}(t), U_{-j}'(t), U_{-j}''(t)\right)$$
$$U_i' = u_n'\gamma_i + c_i h u_n'' + \frac{(c_i h)^2}{2}f(t_n, u_n, u_n') + h^3 \sum_{j=1}^{s} \bar{A}_{i,j} g\left(t_n + c_i h, U_j(t), U_j'(t), U_j''(t)\right)$$

$$U'_{-i} = u'_{n-1}\gamma_i + c_i h u''_{n-1} + \frac{(c_i h)^2}{2} f(t_{n-1}, u_{n-1}, u'_{n-1})$$
$$+h^3 \sum_{j=1}^{s} \bar{A}_{i,j} g\left(t_n + c_i h, U_{-j}(t), U'_{-j}(t), U''_{-j}(t)\right)$$
$$U''_i = u''_n + c_i h f(t_n, u_n, u'_n)\hat{\gamma}_i + h^2 \sum_{j=1}^{s} \hat{A}_{i,j} g\left(t_n + c_i h, U_j(t), U'_j(t), U''_j(t)\right)$$
$$U''_{-i} = u''_{n-1} + c_i h f(t_{n-1}, u_{n-1}, u'_{n-1})\hat{\gamma}_i$$
$$+h^2 \sum_{j=1}^{s} \hat{A}_{i,j} g\left(t_{n-1} + c_i h, U_{-j}(t), U'_{-j}(t), U''_{-j}(t)\right)$$

for i = 1, 2, ..., s. (4)

The parameters for the TDIRKT method are $c_i, A_{i,j}, \bar{A}_{i,j}, \hat{A}_{i,j}, b_i, b'_i, b''_i, \gamma_i$ and $\hat{\gamma}_i$ for i = 1, 2, ..., s. TDIRKT method is explicit if all $A_{i,j}, \bar{A}_{i,j}, \hat{A}_{i,j} = 0$ and $i \le j$ and elsewhere for implicit TDIRKT method. The general TDIRKT method is modified into the form of Butcher tableau, which is exhibited in Table 1.

Table 1

General formulation for TDIRKT methods in butcher tableau

$\mathbf{c}$	$\boldsymbol{A}$	$\boldsymbol{\bar{A}}$	$\boldsymbol{\hat{A}}$
$\boldsymbol{b''_{-1}}$	$\boldsymbol{b^T}$	$\boldsymbol{b'^T}$	$\boldsymbol{b''^T}$

Order Conditions of TDIRKT Method

Order conditions for explicit TDIRKT method up to order 7 are shown in Equations 5 to 17.

The order conditions of u:

Fifth order: $\sum_{i=2}^{s} b_i = \frac{31}{720}$, (5)

Sixth order: $\sum_{i=2}^{s} b_i c_i = \frac{1}{120}$, (6)

Seventh order: $\sum_{i=2}^{s} b_i c_i^2 = \frac{1}{756}$, (7)

The order conditions of u':

Fourth order: $\sum_{i=2}^{s} b'_i = \frac{31}{720}$, (8) Fifth order: $\sum_{i=2}^{s} b_i' c_i = \frac{1}{120}$, (9)

Sixth order: $\sum_{i=2}^{s} b_i' c_i^2 = \frac{1}{756}$, (10)

Seventh order: $\sum_{i=2}^{s} b_i' c_i^3 = \frac{1}{252}$, $\sum_{i=2,j<i}^{s} b_i' \hat{A}_{i,j} = \frac{1}{1512}$, (11)

The order conditions of u'':

Second order: $b_1'' - b_{-1}'' = 0$, (12)

Third order: $\sum_{i=2}^{s} b_i'' + b_{-1}'' = \frac{5}{12}$, (13)

Fourth order: $\sum_{i=2}^{s} b_i'' c_i = \frac{1}{6}$, (14)

Fifth order: $\sum_{i=2}^{s} b_i'' c_i^2 = \frac{31}{360}$, (15)

Sixth order: $\sum_{i=2}^{s} b_i'' c_i^3 = \frac{1}{20}$, $\sum_{i=2,j<i}^{s} b_i'' \hat{A}_{i,j} = \frac{1}{120}$, (16)

Seventh order: $\sum_{i=2}^{s} b_i'' c_i^4 = \frac{1}{63}$, $\sum_{i=2,j<i}^{s} b_i'' \bar{A}_{i,j} = \frac{1}{1512}$,

$\sum_{i=2,j<i}^{s} b_i'' \hat{A}_{i,j} c_j = \frac{1}{1512}$, $\sum_{i=2,j<i}^{s} b_i'' c_i \hat{A}_{i,j} = \frac{1}{378}$, (17)

The coefficients of the improved TDRKT method with three-stages sixth order are shown in Table 2 in the form of Butcher tableau (Equation 18).

Table 2

The improved TDRKT method with a three-stage sixth order

0	**0**	**0**	**0**			**0**			**0**			
c_2	γ_2	$\hat{\gamma}_2$	$A_{2,1}$	0		$\bar{A}_{2,1}$	0		$\hat{A}_{2,1}$	0		
c_3	γ_3	$\hat{\gamma}_3$	$A_{3,1}$	$A_{3,2}$	0	$\bar{A}_{3,1}$	$\bar{A}_{3,2}$	0	$\hat{A}_{3,1}$	$\hat{A}_{3,2}$	0	
b_{-1}''			b_1	b_2	b_3	b_1'	b_2'	b_3'	b_1''	b_2''	b_3''	

where

$$c_1 = 0, c_2 = \frac{5-\sqrt{5}}{10}, b_1 = \frac{1}{40}, b_2 = \frac{1}{60}, b_2' = \frac{1}{12} + \frac{\sqrt{4790}}{1440}, b_3' = \frac{1}{12} - \frac{\sqrt{4790}}{1440},$$

$$b_{-1}'' = -\frac{7}{558}, b_1'' = -\frac{7}{558}, b_2'' = \frac{479}{2232} - \frac{\sqrt{4790}}{59396}, b_3'' = \frac{479}{2232} + \frac{\sqrt{4790}}{59396},$$

$$\hat{A}_{3,2} = -\frac{172440}{3545129}\hat{A}_{2,1}\left(\frac{12}{31} - \frac{\sqrt{4790}}{310}\right) - \frac{107836199}{109898999}\hat{A}_{2,1} - \hat{A}_{3,1} + \frac{688323}{17725645} - \frac{54\sqrt{4790}}{17725645},$$

$$\gamma_2 = 1, \gamma_3 = 1, \hat{\gamma}_2 = 1, \hat{\gamma}_3 = 1. \tag{18}$$

Exponentially-Fitted TDIRKT Method

To develop an exponentially-fitted two-derivative improved Runge-Kutta-type method, we integrate $e^{\omega x}$ and $e^{-\omega x}$ at every stage. The relations $\sinh(v) = (e^v - e^{-v})/2$ and $\cosh(v) = (e^v + e^{-v})/2$, we get the following Equations 19 to 21:

$$e^{\pm c_i v} = 1 \pm c_i v + \frac{1}{2}(c_i v)^2 \pm \frac{1}{6}(c_i v)^3 + v^4 \sum_{i=2}^{s} A_{i,j} e^{\pm c_j v}, \tag{19}$$

$$e^{\pm c_i v} = \gamma_i \pm c_i v + \frac{1}{2}(c_i v)^2 + v^3 \sum_{j=1}^{s} \bar{A}_{i,j} e^{\pm c_j v}, \tag{20}$$

$$e^{\pm c_i v} = 1 \pm \hat{\gamma}_i c_i v + v^2 \sum_{j=1}^{s} \hat{A}_{i,j} e^{\pm c_j v}, \tag{21}$$

Similarly, we integrate $e^{\omega x}$ and $e^{-\omega x}$ corresponding to u, u' and $u,''$ we obtain Equations 22 to 24.

$$e^{\pm v} = 1 \pm \frac{3}{2}v \mp \frac{v}{2}e^{\mp v} + \frac{5}{12}v^2\left(1 - e^{\mp v}\right) \pm \frac{1}{6}v^3\left(1 - e^{\mp v}\right) + v^4 \sum_{i=2}^{s} b_i\left[e^{\pm c_i v} - e^{\pm v(c_i - 1)}\right] \tag{22}$$

$$e^{\pm v} = 1 \pm \frac{3}{2}v \mp \frac{v}{2}e^{\mp v} + \frac{5}{12}v^2\left(1 - e^{\mp v}\right) \mp v^3 \sum_{i=2}^{s} b_i'\left[e^{\pm c_i v} - e^{\pm v(c_i - 1)}\right] \tag{23}$$

$$e^{\pm v} = 1 \pm \frac{3}{2}v \mp \frac{v}{2}e^{\mp v} + v^2 \sum_{i=2}^{s} b_i''\left[e^{\pm c_i v} - e^{\pm v(c_i - 1)}\right] \tag{24}$$

where $v = \omega h, \omega \in \mathrm{R}$. The relation and are substituted in Equations 22 to 24.

Here, we obtain hyperbolic functions of v as in Equation 25.

$$\cosh(v) = 1 + \frac{v}{2}\sinh(v) + \frac{5}{12}v^2(1 - \cosh(v)) + \frac{1}{6}v^3\sinh(v)$$

$$+v^4 \sum_{i=2}^{s} b_i[\cosh(vc_i) - \cosh(v(c_i - 1))],$$

$$\sinh(v) = \frac{3}{2}v - \frac{v}{2}\cosh(v) + \frac{5}{12}v^2\sinh(v) + \frac{1}{6}v^3(1-\cosh(v))$$
$$+v^4\sum_{i=2}^{s} b_i[\sinh(vc_i) - \sinh(v(c_i-1))],$$
$$\cosh(v) = 1 + \frac{v}{2}\sinh(v) + \frac{5}{12}v^2(1-\cosh(v)),$$
$$+v^3\sum_{i=2}^{s} b_i'[\cosh(vc_i) - \cosh(v(c_i-1))]$$
$$\sinh(v) = \frac{3}{2}v - \frac{v}{2}\cosh(v) + \frac{5}{12}v^2\sinh(v) + v^3\sum_{i=2}^{s} b_i'[\sinh(vc_i) - \sinh(v(c_i-1)$$
$$\cosh(v) = 1 + \frac{v}{2}\sinh(v) + v^2\sum_{i=2}^{s} b_i''[\cosh(vc_i) - \cosh(v(c_i-1))],$$
$$\sinh(v) = \frac{3}{2}v - \frac{v}{2}\cosh(v) + v^2\sum_{i=2}^{s} b_i''[\sinh(vc_i) - \sinh(v(c_i-1))], \tag{25}$$

Solving Equations 19 to 21, the coefficients $A_{i,j}, \bar{A}_{i,j}, \hat{A}_{i,j}, \gamma_i$ and $\hat{\gamma}_i$ can be determined as Equations 26 to 30:

$$A_{i,i-1} = \frac{\cosh(vc_i) - 1 - \frac{1}{2}(c_i v)^2 - v^4\sum_{j=1}^{i-2} A_{i,j}\cosh(vc_j)}{v^4\cosh(vc_{i-1})}, \tag{26}$$

$$\bar{A}_{i,i-1} = \frac{\sinh(vc_i) - vc_i - v^3\sum_{j=1}^{i-2} \bar{A}_{i,j}\cosh(vc_j)}{v^3\cosh(vc_{i-1})}, \tag{27}$$

$$\hat{A}_{i,i-1} = \frac{\cosh(vc_i) - 1 - v^2\sum_{j=1}^{i-2} \hat{A}_{i,j}\cosh(vc_j)}{v^2\cosh(vc_{i-1})}, \tag{28}$$

$$\gamma_i = \cosh(vc_i) - \frac{1}{2}(c_i v)^2 - v^3\sum_{j=1}^{i-1} \bar{A}_{i,j}\sinh(vc_j), \tag{29}$$

$$\hat{\gamma}_i = \frac{\sinh(vc_i) - v^2\sum_{j=1}^{i-1} \hat{A}_{i,j}\sinh(vc_j)}{vc_i}. \tag{30}$$

Then, Equations 26 to 30 are integrated by replacing with Equation 18, and we get the terms as in Equation 31.

$$A_{2,1} = \frac{\cosh(vc_2) - 1 - \frac{1}{2}(c_2 v)^2}{v^4},$$

$$A_{3,2} = \frac{\cosh(vc_3) - 1 - \frac{1}{2}(c_3 v)^2 - v^4 A_{3,1}}{v^4\cosh(vc_2)},$$

$$\bar{A}_{2,1} = \frac{\sinh(vc_2) - vc_2}{v^3},$$

$$\bar{A}_{3,2} = \frac{\sinh(vc_3) - vc_3 - v^3\bar{A}_{3,1}}{v^3\cosh(vc_2)},$$

$$\hat{A}_{2,1} = \frac{\cosh(vc_i) - 1}{v^2},$$

$$\hat{A}_{3,2} = \frac{\sinh(vc_3) - 1 - v^2\hat{A}_{3,1}}{v^2\cosh(vc_2)},$$

$$\gamma_2 = \cosh(vc_2) - \frac{1}{2}(c_2v)^2,$$

$$\gamma_3 = \cosh(vc_3) - \frac{1}{2}(c_3v)^2 - v^3\bar{A}_{3,2}\sinh(vc_2),$$

$$\hat{\gamma}_2 = \frac{\sinh(vc_2)}{vc_2},$$

$$\hat{\gamma}_3 = \frac{\sinh(vc_3) - v^2\hat{A}_{3,2}\sinh(vc_2)}{vc_3}, \tag{31}$$

Equation 31 can be further modified through Taylor series expansion, yield Equation 32.

$$A_{2,1} = \frac{6441601}{2216450400} - \frac{1919\sqrt{4790}}{46176050} + \left(\frac{22950627839}{639002650320000} - \frac{4602241\sqrt{4790}}{8875036810000}\right)v^2$$
$$+ \left(\frac{82135555848961}{343885666296211200000} - \frac{12361432319\sqrt{4790}}{3582142357252200000}\right)v^4$$
$$+ \left(\frac{294040876575811199}{29742671277959306688000000} - \frac{35404382065921\sqrt{4790}}{247855593982994222400000}\right)v^6$$
$$+ \left(\frac{105267497362477507008 1}{377291733695169397198617600000000} - \frac{15089188630912457\sqrt{4790}}{374297354856318846427200000000}\right)v^8,$$

$$A_{3,2} = \frac{6441601}{2216450400} + \frac{1919\sqrt{4790}}{46176050} + \left(\frac{1816462073}{319501325160000} + \frac{3841\sqrt{4790}}{2218759202500}\right)v^2$$
$$+ \left(\frac{214076712633923}{343885666296211200000} - \frac{12449882867\sqrt{4790}}{3582142357252200000}\right)v^4$$
$$- \left(\frac{710975672934795007}{106223825992711809600000000} - \frac{1142430575083613\sqrt{4790}}{123927796991497111200000 0}\right)v^6$$
$$+ \left(\frac{3731786017565356409130181}{377291733695169397198617600000000} - \frac{373072793565778149857\sqrt{4790}}{262008148399423192499040000000 0}\right)v^8,$$

$$\bar{A}_{2,1} = \frac{2877}{148955} - \frac{4799\sqrt{4790}}{17874600} + \left(\frac{2023681}{5725830200} - \frac{17495041\sqrt{4790}}{3435498120000}\right)v^2$$

$$+ \left(\frac{36177468473}{11555297926620000} - \frac{62713990079\sqrt{4790}}{1386635751194400000}\right)v^4$$

$$+ \left(\frac{2055628007863}{126910186371220800000} - \frac{224539256163841\sqrt{4790}}{95944100896642924800000}\right)v^6$$

$$+\left(\frac{463627866671572789}{8451875754820102983840000000} - \frac{803863978325073599\sqrt{4790}}{1014225090578412358060800000000}\right)v^8,$$

$$\bar{A}_{3,2} = \frac{1586517831}{86745135890} - \frac{4799\sqrt{4790}}{17874600} + \left(\frac{111041410223}{833620755902900} - \frac{374850995583\sqrt{4790}}{333448302361160000}\right)v^2$$

$$+ \left(-\frac{1010573345229507}{320443818569074760000} + \frac{912680614276573\sqrt{4790}}{115359774684866913600000}\right)v^4$$

$$+ \left(\frac{24054389059591111633}{3695358115738570132320000} - \frac{378876794375477242093\sqrt{4790}}{399098676499765574290560000000}\right)v^6 -$$

$$-\left(\frac{48593533395189696276329848 9}{492201746082552553346309472000000 0}\right)$$

$$- \left(\frac{14058830908639538794050317 3\sqrt{4790}}{98440349216510510669261894400000000}\right)v^8,$$

$$\hat{A}_{2,1} = \frac{1919}{19220} - \frac{6\sqrt{4790}}{4805} + \left(\frac{6441601}{2216450400} - \frac{1919\sqrt{4790}}{46176050}\right)v^2$$

$$+ \left(\frac{22950627839}{639002650320000} - \frac{4602241\sqrt{4790}}{8875036810000}\right)v^4$$

$$+ \left(\frac{82135555848961}{343885666296211200000} - \frac{12361432319\sqrt{4790}}{3582142357252200000}\right)v^6$$

$$+ \left(\frac{294040876575811199}{297426712779593066880000000} - \frac{35404382065921\sqrt{4790}}{247855593982994222400000 0}\right)v^8,$$

$$\hat{A}_{3,2} = \frac{28631109}{793997420} + \frac{6\sqrt{4790}}{4805} + \left(\frac{24812903941}{3662551298976} - \frac{72658291\sqrt{4790}}{1907578801550}\right)v^2$$

$$- \left(\frac{19126585411486231}{2639783848736952 0000} - \frac{3904934862021\sqrt{4790}}{366636645657910000}\right)v^4$$

$$+ \left(\frac{314869203983050706341}{28412521520725561766400 00} - \frac{236564868394367731\sqrt{4790}}{1479818829204456342000 00}\right)v^6$$

$$- \left(\frac{2055056665018462609684485 31}{122869949316377691858796800000 00} - \frac{24742878882947640783241\sqrt{4790}}{102391624430314743215664000000}\right)v^8,$$

$$\gamma_2 = 1 + \left(\frac{6441601}{2216450400} - \frac{1919\sqrt{4790}}{46176050}\right)v^4 + \left(\frac{22950627839}{639002650320000} - \frac{4602241\sqrt{4790}}{8875036810000}\right)v^6 + \left(-\frac{82135555848961}{343885666296211200000} - \frac{12361432319\sqrt{4790}}{358214235725220000}\right)v^8$$

$$\gamma_3 = 1 + \left(-\frac{3231057493}{129076762204320} - \frac{90673209\sqrt{4790}}{26890992125900}\right)v^4 + \left(-\frac{55410348606113}{886019774845368000} + \frac{122935235569\sqrt{4790}}{8875036815168448686597980000000}\right)v^6 + \left(-\frac{3351360448784910563}{20435160087033567552000000} - \frac{746512915113451\sqrt{4790}}{2980127512692395268000}\right)v^8$$

$$\hat{\gamma}_2 = 1 + \left(\frac{4799\sqrt{4790}-345240}{5766(\sqrt{4790}-120)}\right)v^2 + \left(\frac{17495041\sqrt{4790}-1214208600}{11082252000(\sqrt{4790}-120)}\right)v^4 + \left(\frac{62713990079\sqrt{4790}-4341296216760}{4473018552240000(\sqrt{4790}-120)}\right)v^6 + \left(\frac{224539256163841\sqrt{4790}-15540547739444280}{3094970996665900800000(\sqrt{4790}-120)}\right)v^8$$

$$\hat{\gamma}_3 = 1 + \left(\frac{4(4548889\sqrt{4790}-1137646755)}{59549806(\sqrt{4790}+120)}\right)v^2 + \left(\frac{644664336581\sqrt{4790}-46082914655200}{7630315206200(\sqrt{4790}+120)}\right) + \left(\frac{117946514812127927\sqrt{4790}-8193004367907021360}{9239243470579332000(\sqrt{4790}+120)}\right)v^6 + \left(\frac{1232616719282080083158 3\sqrt{4790}-85330343632743040562208 0}{63928173421632513974400000(\sqrt{4790}+120)}\right)v^8 \quad (32)$$

Afterwards, the coefficients in Equation 25 are utilised to obtain parameters $b_2, b_3, b'_2, b'_3, b''_2$ and b''_3 through Taylor series expansion, yielding Equation 33.

$$b_2 = \frac{31}{1440} + \frac{31\sqrt{4790}}{114960} + \left(\frac{2377\sqrt{4790}}{1496779200} + \frac{2651}{18748800}\right)v^2 + \left(\frac{12453977\sqrt{4790}}{2589128660160000} - \frac{14221}{20591539200}\right)v^4 + \left(\frac{152021375417\sqrt{4790}}{16421807439930816000000} + \frac{184288690451}{6856704567820800000}\right)v^6 + \left(-\frac{429376762287585683\sqrt{4790}}{17233241789152677480192000000 0} - \frac{2048157093308653}{26165483832458041344000000}\right)v^8,$$

$$b_3 = \frac{31}{1440} - \frac{31\sqrt{4790}}{114960} + \left(-\frac{2377\sqrt{4790}}{1496779200} + \frac{2651}{18748800}\right)v^2 - \left(\frac{12453977\sqrt{4790}}{2589128660160000} - \frac{14221}{20591539200}\right)v^4 + \left(-\frac{152021375417\sqrt{4790}}{16421807439930816000000} + \frac{184288690451}{6856704567820800000}\right)v^6 + \left(-\frac{429376762287585683\sqrt{4790}}{17233241789152677480192000000 0} - \frac{2048157093308653}{26165483832458041344000000}\right)v^8,$$

$$
\begin{aligned}
b'_{2'} = {} & \frac{1}{12} - \frac{\sqrt{4790}}{1440} + \left(\frac{241\sqrt{4790}}{96566400}\right)v^2 + \left(\frac{518389\sqrt{4790}}{18560062080000} + \frac{1973}{193737600}\right)v^4 \\
& + \left(-\frac{76233665681\sqrt{4790}}{105947144773747200000} - \frac{10186361}{41890912560000}\right)v^6 \\
& + \left(-\frac{204301546417014479\sqrt{4790}}{1111822050913075966464000000} + \frac{294637773487}{47235075911654400000}\right)v^8, \\
b'_{3'} = {} & \frac{1}{12} - \frac{\sqrt{4790}}{1440} - \left(\frac{241\sqrt{4790}}{96566400}\right)v^2 + \left(-\frac{518389\sqrt{4790}}{18560062080000} + \frac{1973}{193737600}\right)v^4 \\
& + \left(\frac{76233665681\sqrt{4790}}{105947144773747200000} - \frac{10186361}{41890912560000}\right)v^6 \\
& + \left(-\frac{204301546417014479\sqrt{4790}}{1111822050913075966464000000} + \frac{294637773487}{47235075911654400000}\right)v^8, \\
b'_{2'} = {} & \frac{479}{2232} - \frac{\sqrt{4790}}{59396} + \left(-\frac{351\sqrt{4790}}{1663088000} + \frac{1907}{37497600}\right)v^4 + \left(-\frac{2412191\sqrt{4790}}{647282165040000} - \frac{492499}{386091360000}\right)v^6 \\
& + \left(-\frac{3273299154193\sqrt{4790}}{328436148798616320000000} + \frac{977694255779}{274268182712832000000}\right)v^8, \\
b'_{3'} = {} & \frac{479}{2232} + \frac{\sqrt{4790}}{59396} + \left(\frac{351\sqrt{4790}}{1663088000} + \frac{1907}{37497600}\right)v^4 + \left(-\frac{2412191\sqrt{4790}}{647282165040000} - \frac{492499}{386091360000}\right)v^6 \\
& + \left(\frac{3273299154193\sqrt{4790}}{328436148798616320000000} + \frac{977694255779}{274268182712832000000}\right)v^8, \qquad (33)
\end{aligned}
$$

where γ_i , $\hat{\gamma}_i = 1$.

Trigonometrically-Fitted Improved TDRKT Method

Trigonometrically-fitted improved TDRKT method can be derived by substituting $v = wh$ with iwh and solving Equations 19 to 21 to obtain the coefficients.

$$A_{i,i-1} = \frac{\cos(vc_i) - 1 + \frac{1}{2}(c_i v)^2 - v^4 \sum_{j=1}^{i-2} A_{i,j}\cos(vc_j)}{v^4 \cos(vc_{i-1})}, \qquad (34)$$

$$\bar{A}_{i,i-1} = \frac{vc_i - \sin(vc_i) - v^3 \sum_{j=1}^{i-2} \bar{A}_{i,j}\cos(vc_j)}{v^3 \cos(vc_{i-1})}, \qquad (35)$$

$$\hat{A}_{i,i-1} = \frac{1 - \cosh(vc_i) + v^2 \sum_{j=1}^{i-2} \hat{A}_{i,j}\cos(vc_j)}{v^2 \cos(vc_{i-1})}, \qquad (36)$$

$$\gamma_i = \cos(vc_i) + \frac{1}{2}(c_i v)^2 + v^3 \sum_{j=1}^{i-1} \bar{A}_{i,j}\sin(vc_j), \qquad (37)$$

$$\hat{\gamma}_i = \frac{\sin(vc_i) + v^2 \sum_{j=1}^{i-1} \hat{A}_{i,j}\sin(vc_j)}{vc_i}. \qquad (38)$$

Later, Equations 34 to 38 are solved by substituting Equation 18.

$$A_{2,1} = \frac{\cos(vc_2) - 1 + \frac{1}{2}(c_2 v)^2}{v^4},$$

$$A_{3,2} = \frac{\cos(vc_3) - 1 + \frac{1}{2}(c_3 v)^2 - v^4 A_{3,1}}{v^4 \cos(vc_2)},$$

$$\bar{A}_{2,1} = \frac{vc_2 - \sin(vc_2)}{v^3},$$

$$\bar{A}_{3,2} = \frac{vc_3 - \sin(vc_3) - v^3 \bar{A}_{3,1}}{v^3 \cos(vc_2)},$$

$$\hat{A}_{2,1} = \frac{1 - \cos(vc_i)}{v^2},$$

$$\hat{A}_{3,2} = \frac{1 - \sin(vc_3) + v^2 \hat{A}_{3,1}}{v^2 \cos(vc_2)},$$

$$\gamma_2 = \cos(vc_2) + \frac{1}{2}(c_2 v)^2,$$

$$\gamma_3 = \cos(vc_3) + \frac{1}{2}(c_3 v)^2 + v^3 \bar{A}_{3,2} \sin(vc_2),$$

$$\hat{\gamma}_2 = \frac{\sin(vc_2)}{vc_2},$$

$$\hat{\gamma}_3 = \frac{\sinh(vc_3) + v^2 \hat{A}_{3,2} \sin(vc_2)}{vc_3}, \qquad (39)$$

Equation 39 is then modified through Taylor series expansion, generating Equation 40:

$$A_{2,1} = \frac{6441601}{2216450400} - \frac{1919\sqrt{4790}}{46176050} - \left(\frac{22950627839}{639002650320000} - \frac{4602241\sqrt{4790}}{8875036810000}\right) v^2$$

$$+ \left(\frac{82135555848961}{343885666296211200000} - \frac{12361432319\sqrt{4790}}{3582142357252200000}\right) v^4$$

$$- \left(\frac{294040876575811199}{29742671277959306688000000} - \frac{35404382065921\sqrt{4790}}{247855593982994222400000}\right) v^6$$

$$+ \left(\frac{1052674973624775070081}{3772917336951693971986176000000000} - \frac{15089188630912457\sqrt{4790}}{374297354856318846427200000000}\right) v^8,$$

$$A_{3,2} = \frac{6441601}{2216450400} + \frac{1919\sqrt{4790}}{46176050} + \left(\frac{1816462073}{319501325160000} + \frac{3841\sqrt{4790}}{2218759202500}\right)v^2$$

$$+\left(\frac{214076712633923}{343885666296211200000} - \frac{12449882867\sqrt{4790}}{3582142357252200000}\right)v^4$$

$$+\left(\frac{710975672934795007}{1062238259927118096000000} - \frac{1142430575083613\sqrt{4790}}{1239277969914971112000000}\right)v^6$$

$$+\left(\frac{37317860175653564091301 81}{377291733695169397198617600000000} - \frac{373072793565778149857\sqrt{4790}}{262008148399423192499040000000 0}\right)v^8,$$

$$\bar{A}_{2,1} = \frac{2877}{148955} - \frac{4799\sqrt{4790}}{17874600} - \left(\frac{2023681}{5725830200} - \frac{17495041\sqrt{4790}}{3435498120000}\right)v^2$$

$$+\left(\frac{36177468473}{11555297926620000} - \frac{62713990079\sqrt{4790}}{1386635751194400000}\right)v^4$$

$$-\left(\frac{2055628007863}{126910186371220800000} - \frac{224539256163841\sqrt{4790}}{959441008966429248000000}\right)v^6$$

$$+\left(\frac{463627866671572789}{845187575482010298384000000 0} - \frac{803863978325073599\sqrt{4790}}{1014225090578412358060800000000}\right)v^8,$$

$$\bar{A}_{3,2} = \frac{1586517831}{86745135890} - \frac{4799\sqrt{4790}}{17874600} - \left(\frac{111041410223}{833620755902900} - \frac{374850995583\sqrt{4790}}{333448302361160000}\right)v^2$$

$$+\left(-\frac{1010573345229507}{320443818569074760000} + \frac{912680614276573\sqrt{4790}}{115359774684866913600000}\right)v^4$$

$$-\left(\frac{240543890595911116 33}{369535811573857013232000000} - \frac{378876794375477242093\sqrt{4790}}{399098676499765574290560000 00}\right)v^6$$

$$-\left(\frac{485935333951896962763298489}{492201746082552553346309472000000 00} - \frac{140588309086395387940503173\sqrt{4790}}{984403492165105106692618944000000000}\right)v^8,$$

$$\hat{A}_{2,1} = \frac{1919}{19220} - \frac{6\sqrt{4790}}{4805} - \left(\frac{6441601}{2216450400} - \frac{1919\sqrt{4790}}{46176050}\right)v^2$$

$$+ \left(\frac{22950627839}{639002650320000} - \frac{4602241\sqrt{4790}}{8875036810000}\right)v^4$$

$$- \left(\frac{82135555848961}{343885666296211200000} - \frac{12361432319\sqrt{4790}}{3582142357252200000}\right)v^6$$

$$+ \left(\frac{294040876575811199}{297426712779593066880000000} - \frac{35404382065921\sqrt{4790}}{2478555939829942224000000}\right)v^8,$$

$$\hat{A}_{3,2} = \frac{28631109}{793997420} + \frac{6\sqrt{4790}}{4805} - \left(\frac{24812903941}{3662551298976} - \frac{72658291\sqrt{4790}}{1907578801550}\right)v^2$$

$$- \left(\frac{19126585411486231}{263978384873695200000} - \frac{3904934862021\sqrt{4790}}{366636645657910000}\right)v^4$$

$$- \left(\frac{314869203983050706341}{2841252152072556176640000} - \frac{236564868394367731\sqrt{4790}}{147981882920445634200000}\right)v^6$$

$$- \left(\frac{205505666501846260968448531}{12286994931637769185879680000000} - \frac{24742878882947640783241\sqrt{4790}}{102391624430314743215664000000}\right)v^8,$$

$$\gamma_2 = 1 + \left(\frac{6441601}{2216450400} - \frac{1919\sqrt{4790}}{46176050}\right)v^4 - \left(\frac{22950627839}{639002650320000} - \frac{4602241\sqrt{4790}}{8875036810000}\right)v^6$$

$$+\left(-\frac{82135555848961}{343885666296211200000} - \frac{12361432319\sqrt{4790}}{3582142357252200000}\right)v^8$$

$$\gamma_3 = 1 + \left(-\frac{3231057493}{129076762204320} - \frac{90673209\sqrt{4790}}{26890992125900}\right)v^4$$

$$-\left(-\frac{55410348606113}{8860197748453680000} + \frac{122935235569\sqrt{4790}}{887503681516844868659798000000}\right)v^6$$

$$+\left(-\frac{3351360448784910563}{2043516008703356755200000} - \frac{746512915113451\sqrt{4790}}{29801275126923952680000}\right)v^8$$

$$\hat{\gamma}_2 = 1 - \left(\frac{4799\sqrt{4790} - 345240}{5766(\sqrt{4790} - 120)}\right)v^2 + \left(\frac{17495041\sqrt{4790} - 1214208600}{11082252000(\sqrt{4790} - 120)}\right)v^4$$

$$-\left(\frac{62713990079\sqrt{4790} - 4341296216760}{4473018552240000(\sqrt{4790} - 120)}\right)v^6$$

$$+\left(\frac{224539256163841\sqrt{4790} - 15540547739444280}{30949709966590080000(\sqrt{4790} - 120)}\right)v^8$$

$$\hat{\gamma}_3 = 1 + \left(\frac{4(4548889\sqrt{4790} - 1137646755)}{59549806(\sqrt{4790} + 120)}\right)v^2 + \left(\frac{644664336581\sqrt{4790} - 46082914655200}{7630315206200(\sqrt{4790} + 120)}\right)v^4$$

$$-\left(\frac{117946514812127927\sqrt{4790} - 8193004367907021360}{9239243470579332000(\sqrt{4790} + 120)}\right)v^6$$

$$+\left(\frac{123261671928208008315 83\sqrt{4790} - 853303436327430405622080}{6392817342163251397440000(\sqrt{4790} + 120)}\right)v^8 \qquad (40)$$

Afterwards, the coefficients in Equation 40 are utilised to obtain parameters $b_2, b_3, b_2', b_3', b_2''$ and b_3'' through Taylor series expansion, yielding Equation 41.

$$b_2 = \frac{31}{1440} + \frac{31\sqrt{4790}}{114960} - \left(\frac{2377\sqrt{4790}}{1496779200} + \frac{2651}{18748800}\right)v^2 + \left(\frac{12453977\sqrt{4790}}{2589128660160000} - \frac{14221}{20591539200}\right)v^4$$

$$-\left(-\frac{152021375417\sqrt{4790}}{1642180743993081600000} + \frac{184288690451}{6856704567820800000}\right)v^6$$

$$+\left(-\frac{429376762287585683\sqrt{4790}}{1723324178915267748019200000000} - \frac{2048157093308653}{2616548383245804134400000}\right)v^8,$$

$$b_3 = -\frac{31}{1440} + \frac{31\sqrt{4790}}{114960} + \left(-\frac{2377\sqrt{4790}}{1496779200} + \frac{2651}{18748800}\right)v^2$$

$$-\left(\frac{12453977\sqrt{4790}}{2589128660160000} - \frac{14221}{20591539200}\right)v^4$$

$$-\left(\frac{152021375417\sqrt{4790}}{164218074399308160000} + \frac{184288690451}{6856704567820800000}\right)v^6$$

$$+\left(-\frac{429376762287585683\sqrt{4790}}{17233241789152677480192000000} - \frac{2048157093308653}{261654838324580413440000}\right)v^8,$$

$$b'_{3'} = \frac{1}{12} - \frac{\sqrt{4790}}{1440} + \left(\frac{241\sqrt{4790}}{96566400}\right)v^2 + \left(-\frac{518389\sqrt{4790}}{18560062080000} + \frac{1973}{193737600}\right)v^4$$

$$-\left(\frac{76233665681\sqrt{4790}}{105947144773747200000} - \frac{10186361}{41890912560000}\right)v^6$$

$$+\left(-\frac{204301546417014479\sqrt{4790}}{1111822050913075966464000000} + \frac{294637773487}{4723507591165440000}\right)v^8,$$

$$b'_{2'} = \frac{479}{2232} - \frac{\sqrt{4790}}{59396} + \left(-\frac{351\sqrt{4790}}{1663088000} + \frac{1907}{37497600}\right)v^4 - \left(\frac{2412191\sqrt{4790}}{647282165040000} - \frac{492499}{386091360000}\right)v^6$$

$$+\left(-\frac{3273299154193\sqrt{4790}}{328436148798616320000000} + \frac{977694255779}{27426818271283200000}\right)v^8,$$

$$b'_{3'} = \frac{479}{2232} + \frac{\sqrt{4790}}{59396} + \left(\frac{351\sqrt{4790}}{1663088000} + \frac{1907}{37497600}\right)v^4 - \left(-\frac{2412191\sqrt{4790}}{647282165040000} - \frac{492499}{386091360000}\right)v^6$$

$$+\left(\frac{3273299154193\sqrt{4790}}{328436148798616320000000} + \frac{977694255779}{27426818271283200000}\right)v^8, \quad (41)$$

where γ_i , $\hat{\gamma}_i = 1$.

As $v \to 0$, the coefficients $b_i, b'_i, b''_i, A_{i,j}, \bar{A}_{i,j}, \hat{A}_{i,j}, \gamma_i$ and $\hat{\gamma}_i$ of the proposed methods will return to the coefficients of the original form. It means that both EFIRKT6 and TFIRKT6 methods have the same error constant as the three-stage, sixth-order improved TDRKT method.

Error Analysis of EFIRKT6 and TFIRKT6 Methods

Local truncation errors (*LTE*) for $u(t), u'(t)$ and $u''(t)$ for EFIRKT6 and TFIRKT6 methods are analysed in this part. Here, Taylor series expansion is applied over the step size, h, for the exact solution, $u\,(t_n + h)$ and its derivatives, $u'(t_n + h)$ and $u''(t_n + h)$. Then, we get local truncation errors of u, u' and u'' as Equation 42:

$$\tau_{n+1}^{(m)} = u_{n+1}^{(m)} - u^{(m)}(t_n + h), \quad m = 0, 1, 2. \tag{42}$$

where u_{n+1}, u'_{n+1} and u''_{n+1} are the approximation solutions for u, u' and u''.

Both EFIRKT6 and TFIRKT6 methods contain algebraic order p if $\text{LTE} = u_{n+1} - u(t_n + h) = u'_{n+1} - u'(t_n + h) = u''_{n+1} - u''(t_n + h) = \mathcal{O}(h^{p+1})$ and $\mathcal{O}(h^q) = 0$, $q = 1, \dots, p$. *LTE* of $u(t), u'(t)$ and $u''(t)$ of proposed methods are shown as in Equations 43 to 45:

$$\begin{aligned} LTE(u) = -\tfrac{11}{2520}h^7(\, & g_{u,u,u}u'^3 + 3g_{u,u,u'}u'^2u'' + 3g_{u,u}u'u'' + 3g_{u,u',u'}u''^2u' \\ & +3g_{u,u'}u'f + 3g_{u,u'}u''^2 + g_uf + g_{u',u',u'}u''^3 + 3g_{u',u'}u''f + g_{u'}g) + \mathcal{O}(h^8) \end{aligned} \tag{43}$$

$$\begin{aligned} & LTE(u') \\ & = \frac{11}{3360}h^7\big(12\, g_{u,u',u'}u'u''f + g_{u',u',u',u'}u''^4 + 3g_{u',u'}u''f^2 + g_ug \\ & + g_{u,u,u,u}u'^4 + 3g_{u,u}u''^2 + 6g_{u,u',u'}u''^2 + g_{u'}(g_uu' + g_{u'}u'') \\ & + 4u'^3g_{u,u,u,u'}u'' + 6u'^2g_{u,u,u}u'' + 6u'^2g_{u,u,u,u'}u''^2 + 6u'^2g_{u,u,u'}f \\ & + 12u'g_{u,u,u'}u''^2 + 4u'g_{u,u}f + 4u'g_{u,u',u',u'}u''^3 + 4u'g_{u,u'}g \\ & + 10u''g_{u,u'}f + 6u''^2g_{u',u',u'}f + 4u''g_{u',u'}g + \mathcal{O}(h^8) \end{aligned} \tag{44}$$

$$\begin{aligned} & LTE(u'') \\ & = \frac{1}{5040}h^7\big(30u'^2u''\, g_{u,u,u',u'}f + 30u'u''^2\, g_{u,u',u',u'}f + 20u'u''g_{u,u',u'}g \\ & + 50u'u''g_{u,u,u'}f + 10g_{u',u'}g + 10u'^3g_{u,u,u,u'}f + 10u'^2g_{u,u,u}f \\ & + 10u'^2g_{u,u,u'}g + 15u'g_{u,u',u'}f^2 + 5u'g_{u,u}g + 10g_{u,u'}f^2 + g_{u,u,u,u,u}u'^5 \\ & + g_{u',u',u',u',u'}u''^5 + 15g_{u,u,u'}u''^3 + 10g_{u,u',u',u'}u''^4 + 10u'^3g_{u,u,u,u}u'' \\ & + g_{u'}\big(g_{u,u}u'^2 + 2u'g_{u,u'}u'' + g_uu'' + g_{u',u'}u''^2 + g_{u'}f\big) \\ & + g_u(g_uu' + g_{u'}u'') + 10g_{u,u,u,u',u'}u''^2 + 30u'^2g_{u,u,u,u'}u''^2 \\ & + 10u'^2g_{u,u,u',u',u'}u''^3 + 15u'g_{u,u,u}u''^2 + 30u'g_{u,u,u',u'}u''^3 \\ & + 5u'g_{u,u',u',u',u'}u''^4 + 5u'g_{u,u'}(g_uu' + g_{u'}u'') + 10u''g_{u,u}f \\ & + 40u''^2g_{u,u',u'}f + 15u''g_{u,u'}g + 10u''^3g_{u',u',u',u'}f + 10u''^2g_{u',u',u'}g \\ & + 15u''g_{u',u',u'}f^2 + 5u''g_{u',u'}g_uu' + g_{u'}u'' + 5u'^4g_{u,u,u,u,u'}u''\big) + \mathcal{O}(h^8) \end{aligned} \tag{45}$$

The order for both EFIRKT6 and TFIRKT6 methods is six since the entire coefficients are up to $h^6 = 0$. Thus, LTE = $\mathcal{O}(h^{p+1}) = \mathcal{O}(h^7)$. By comparison, $p = 6$.

RESULT AND DISCUSSION

EFIRKT6 and TFIRKT6 methods solve $u'''(t) = f(t, u(t), u'(t))$ with exponential or oscillatory solutions and application problems. Problems 1 to 4 are third-order exponential problems, while problems 5 to 7 are third-order trigonometrical problems. The proposed method was also used to test the efficiency in solving the application problem, thin-film flow, in problem 8. The proficiency of EFIRKT6 and TFIRKT6 methods are demonstrated as they are contrasted with the classic Runge-Kutta method and Runge-Kutta direct methods with exponentially-fitted and trigonometrically-fitted techniques. The selected comparative methods contain fitting techniques or have similar order to the proposed methods.
The selected methods as below are compared numerically:

- EFIRKT6: Three-stage sixth-order explicit improved TDRKT method with exponentially-fitting technique.
- TFIRKT6: Three-stage sixth-order explicit improved TDRKT method with trigonometrically-fitting technique.
- EFTDRKT6: Exponentially-fitted explicit TDRKT method with three stages sixth-order, the exponential technique is implemented into the method constructed by Lee et al. (2020)
- TFTDRKT6: Trigonometrically fitted explicit TDRKT method with three stages sixth-order, the trigonometrical technique is implemented into the method constructed by Lee et al. (2020)
- RK6S: Explicit RK method with seven-stage sixth order developed by Al-Shimmary (2017)
- EFRKT5: Four-stage fifth-order exponential-fitted explicit Runge-Kutta type method developed by Ghawadri et al. (2018)
- TFRKT5: Trigonometrically-fitted explicit Runge-Kutta-type method with four stages fifth order, developed by Ghawadri et al. (2018)
- ATDRKT6: Trigonometrically-fitted explicit two-derivative Runge-Kutta method with four stages sixth order, developed by Ahmad et al. (2019)

Problem 1 (Exponential problem)

$$u''' = 2u'(t),$$
$$u(0) = 0, u'(0) = 1, u''(0) = 0, \qquad t\epsilon[0,5],$$

whose analytic solution is $u(t) = \frac{\sqrt{2}e^{\sqrt{2}t}}{4} - \frac{\sqrt{2}e^{-\sqrt{2}t}}{4}$.

Problem 2 (Exponential problem)

$$u''' = 5u'(t) + \sinh(t),$$

$$u(0) = -\frac{1}{4}, u'(0) = 0, u''(0) = \frac{1}{4}, \qquad t\epsilon[0,5],$$

whose analytic solution is $u(t) = -\frac{e^t}{8} - \frac{e^{-t}}{8}$.

Problem 3 (Exponential problem)

$$u_1''' = 8u_3', u_2''' = 8u_1', u_3''' = u_2',$$

$$u_1(0) = 2, u_1' = 4, u_1'' = 8,$$

$$u_2(0) = 4, u_2' = 8, u_2'' = 16,$$

$$u_3(0) = 1, u_3' = 2, u_3'' = 4,$$

whose analytic solution is $u_1(t) = 2e^{2t}, u_2(t) = 4e^{2t}, u_3(t) = e^{2t}, \qquad t\epsilon[0,5]$.

Problem 4 (Exponential problem)

$$u_1''' = u_3' + 1, u_2''' = u_1' + 2, u_3''' = u_2' + 3,$$
$$u_1(0) = 2, u_1' = 3, u_1'' = 5,$$
$$u_2(0) = 1, u_2' = 2, u_2'' = 5,$$
$$u_3(0) = 0, u_3' = 4, u_3'' = 5,$$

whose analytic solution is $u_1(t) = 5e^t - 2t - 3, u_2(t) = 5e^t - 3t - 4, u_3(t) = 5e^t - t - 5, \qquad t\epsilon[0,5]$.

Problem 5 (Trigonometrical problem)

$$u''' = -27u'(t),$$

$$u(0) = 1, u'(0) = 3\sqrt{3}, u''(0) = -27, \qquad t\epsilon[0,10000],$$

whose analytic solution is $u(t) = \cos(3\sqrt{3}t) + \sin(3\sqrt{3}t)$.

Problem 6 (Trigonometrical problem)

$$u''' = -100\,u'(t) + 99\cos(t),$$

$$u(0) = 1, u'(0) = 11, u''(0) = -100, \qquad t\epsilon[0{,}10000],$$

whose analytic solution is $u(t) = \cos(10t) + \sin(10t) + \sin(t)$.
Problem 7 (Trigonometrical problem)

$$u_1''' = 2u_1' + 6u_2', \; u_2''' = -2u_1' - 5u_2',$$

$$u_1(0) = 2, u_1' = 0, u_1'' = -2,$$

$$u_2(0) = -1, u_2' = 0, u_2'' = 1,$$

whose analytic solution is $u_1(t) = 2\cos(t), u_2(t) = -\cos(t), \; t\epsilon[0{,}10000]$.

Problem 8 (Application problem)

Application Problem of Third-Order ODEs-Thin Film Flow

We consider the famous fluid dynamic and engineering problem, the thin film flow of fluid transporting over the solid surface. Usually, thin film flow simulates thermal and mass transfer, gravity and centrifugal force (Kumar & Singh, 2012). According to Duffy and Wilson (1997), thin film flow can describe the dynamic balance between surface tension and viscous force in the thin film layer without gravity. Recently, various direct methods have been developed to solve particular problems (Ghawadri et al., 2018; Lee et al., 2020; Jikantoro et al., 2018; Haweel et al., 2018). The thin film flow problem can be represented by Equation 46:

$$u''' = f(u(t)), \tag{46}$$

where
$u(t)$ implies the cartesian coordinate system in flowing fluid, and we express $f(u(t))$ in various terms:

$$f(u) = -1 + u^{-2},$$

$$f(u) = -1 + (1 + \gamma + \gamma^2)u^{-2} - (\gamma + \gamma^2)u^{-3}$$

$$f(u) = u^{-2} - u^{-3},$$

$$f(u) = u^{-2}.$$

Here, we focus on solving the nonlinear thin film flow problem, which is utilised to demonstrate the fluid-depleting problem on a torrid surface as Equation 47.

$$u''' = u^{-2} - u^{-3}, u(0) = 1, u'(0) = 1, u''(0) = 0 \tag{47}$$

The numerical curve of thin film flow is demonstrated in Figure 1 to exhibit the thin film flow model.

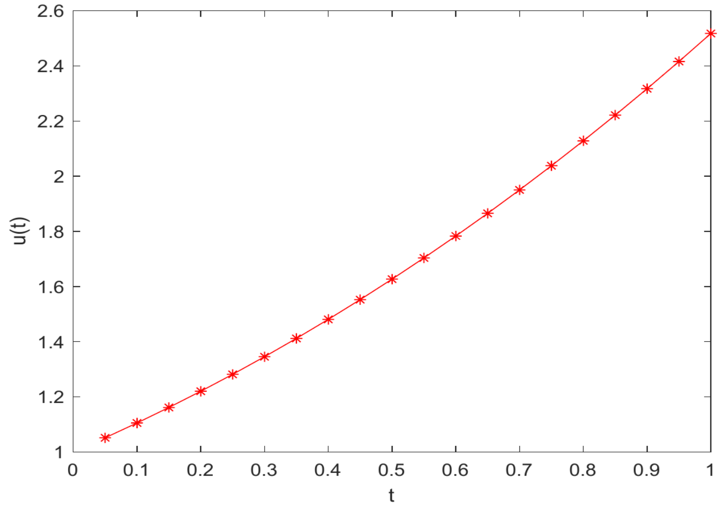

Figure 1. Numerical solution curves for thin film flow

For comparison purposes, the classical order 4 Runge-Kutta method with tremendously low step size, $h = 10^{-6}$ is used to compare the selected methods for obtaining numerical approximation due to the absence of an exact solution in a thin film flow problem. Figure 1 exhibits the numerical results of the Runge-Kutta method with step size $h = 10^{-6}$ in solving problem 8.

Figures 2 to 9 exhibit the performance of selected methods numerically measured through the maximum global truncation error against computational time.

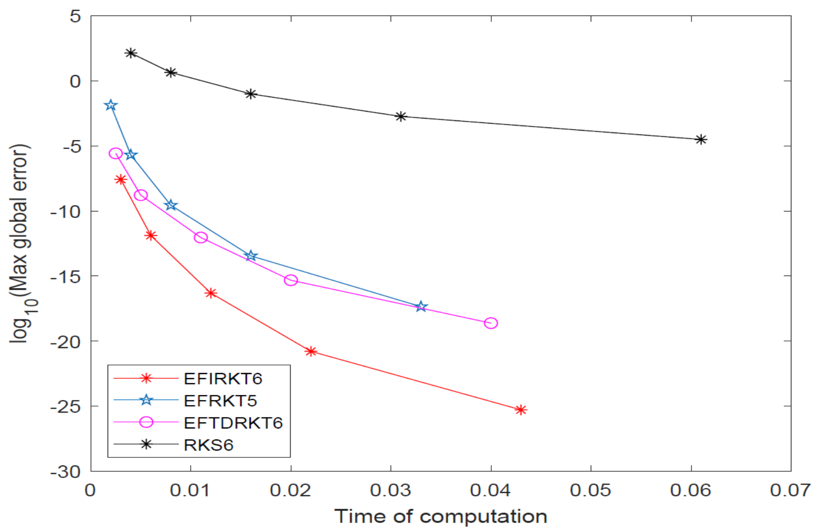

Figure 2. Numerical curves of selected methods of problem 1 with $h = \frac{0.5}{2^i}, i = 0, \ldots, 4.$

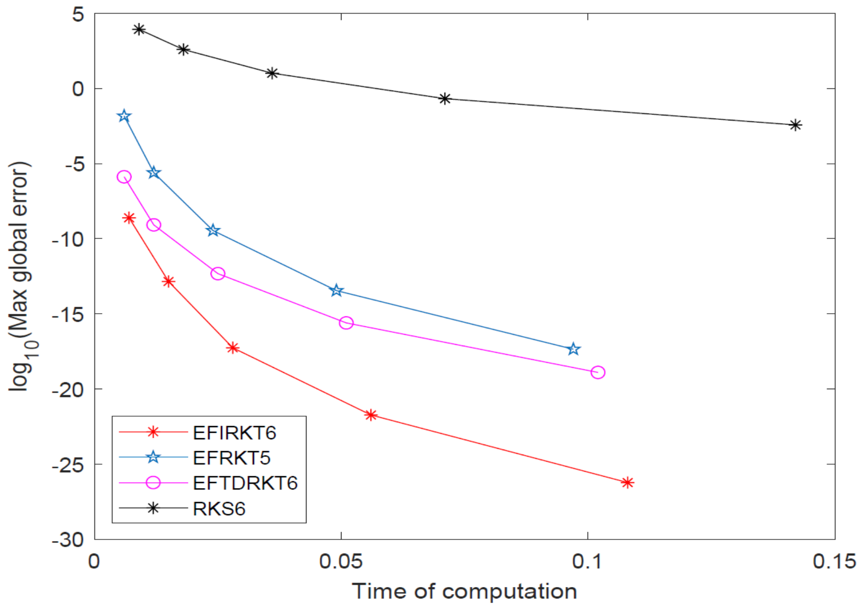

Figure 3. Numerical curves of selected methods of problem 2 with $h = \frac{0.5}{2^i}, i = 0, \ldots, 4.$

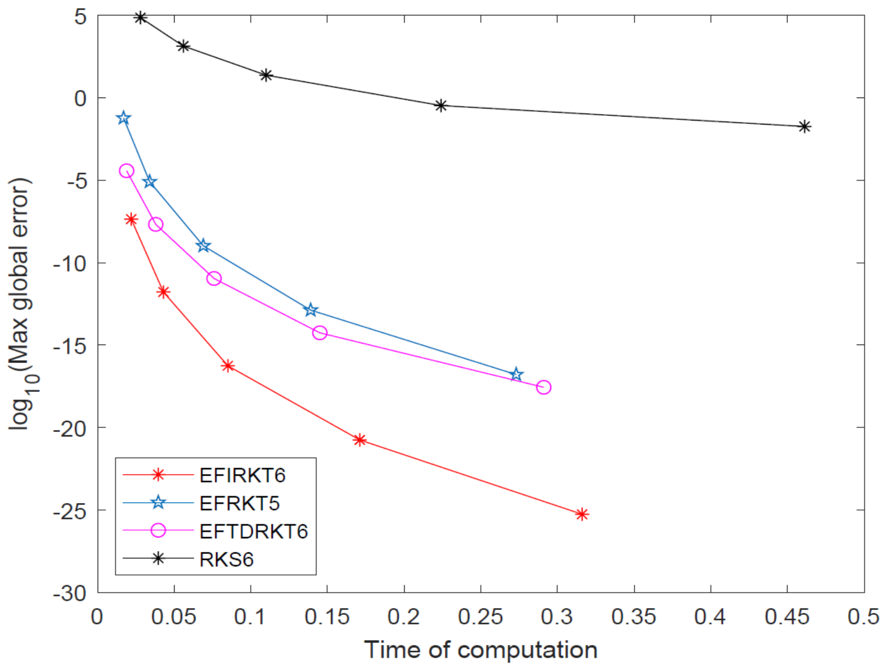

Figure 4. Numerical curves of selected methods of problem 3 with $h = \frac{0.2}{2^i}, i = 0, \ldots, 4.$

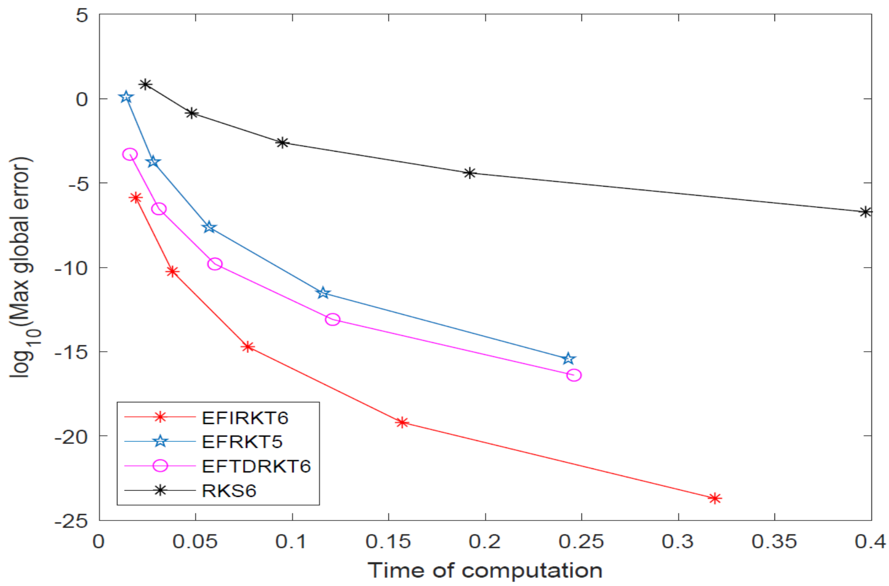

Figure 5. Numerical curves of selected methods of problem 4 with $h = \frac{0.5}{2^i}, i = 0, \ldots, 4.$

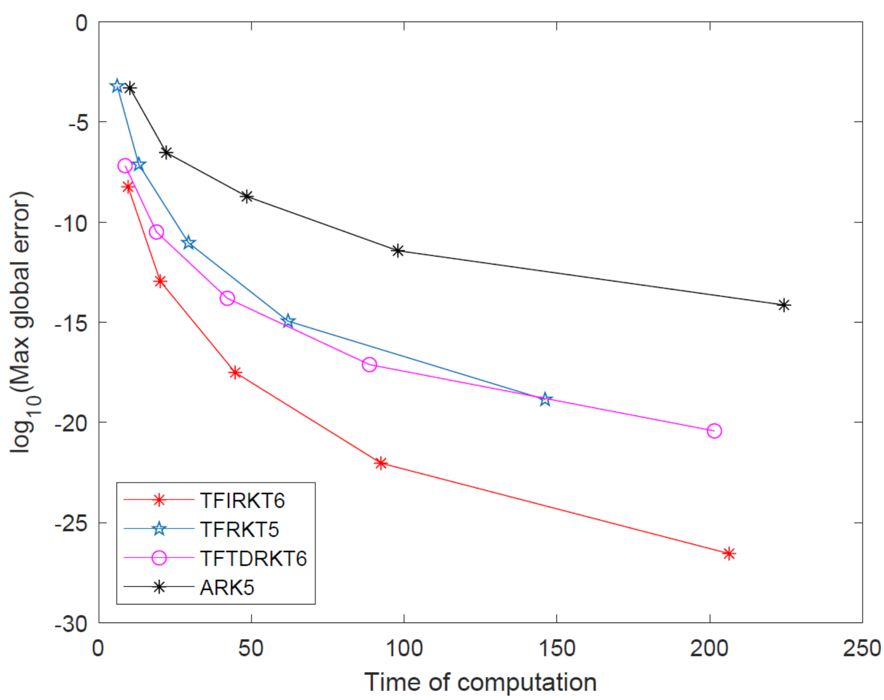

Figure 6. Numerical curves of selected methods of problem 5 with $h = \frac{0.5}{2^i}, i = 0, \ldots, 4.$

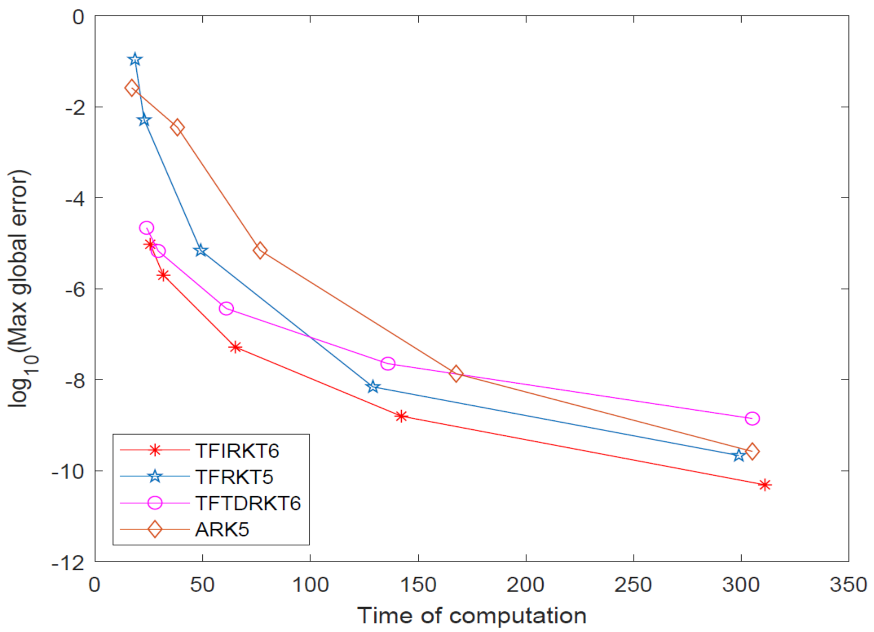

Figure 7. Numerical curves of selected methods of problem 6 with h = 0.25, 0.2, 0.1, 0.05, 0.025

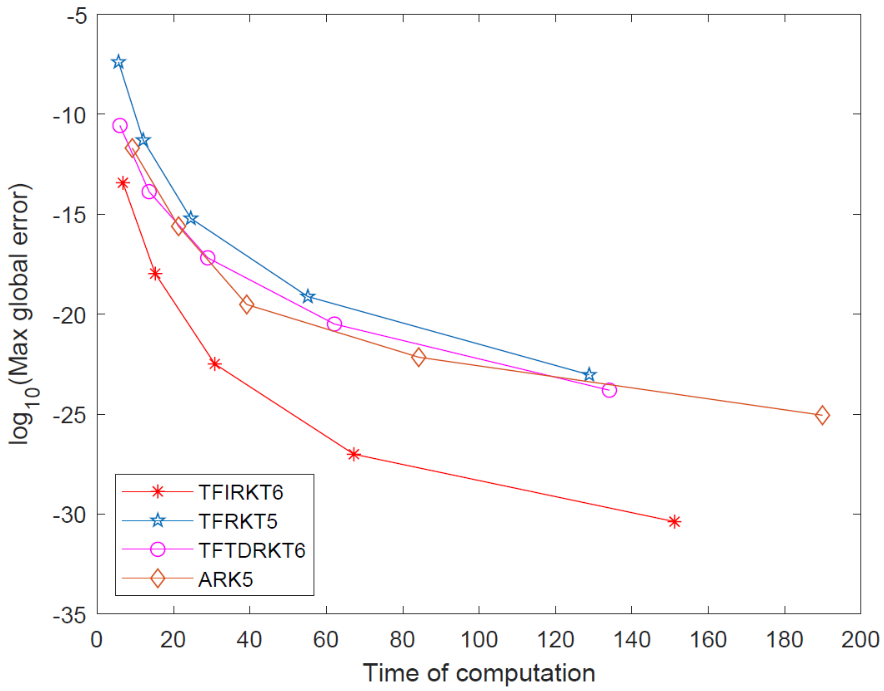

Figure 8. Numerical curves of selected methods of problem 7 with $h = \frac{0.4}{2^i}, i = 0, \ldots, 4.$

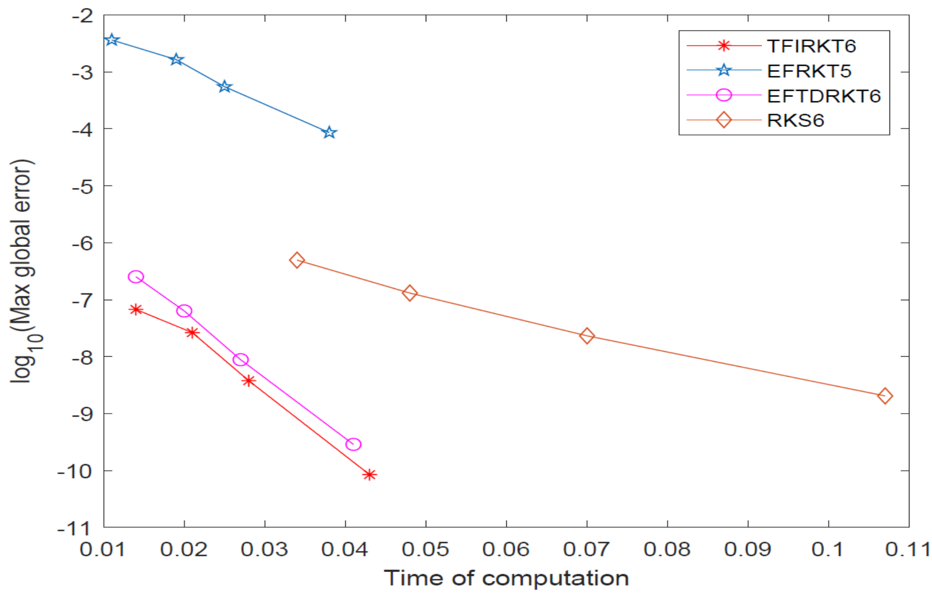

Figure 9. Numerical curves of selected methods of a thin film flow problem

The proposed methods, TDRKT methods fitting techniques are used to contrast with other existing numerical methods in solving third-order ordinary initial value problems with exponential and trigonometrical solutions based on a maximum global error against computation time. The numerical results are plotted in Figures 2 to 9. The numerical approach solves the selected problems by taking step size with smaller values for subsequent approximation and comparing the solutions to illustrate the convergence performance and accuracy curves obtained by all selected methods. Global errors obtained by all methods are getting lesser when the step size becomes smaller. It is because the local truncation error in approximating the numerical problem is reduced when the step size becomes smaller and causes the accuracy for the next approximation to become higher. Improved Runge-Kutta comprises the previous step in the function evaluation, which highly improves the method's accuracy. Hence, the results in Figures 2 to 5 clearly show that the EFIRKT6 method outperforms EFTDRKT6, RKS6 and EFRKT5 methods for solving exponential third-order ODEs by yielding the lowest maximum global error in similar time computation with the same step size. RKS6 acquires a higher number of function evaluations because it requires converting higher-order differential equations into three first-order differential equations and solving them subsequently. Meanwhile, the complexity of function evaluation of the RK6 method is the least compared to the other three selected methods, causing the computation time is not too large comparatively. The complexity of each function evaluation for EFIRKT6 methods is higher than other existing methods because of the inclusion of derivative of *f*-evaluation; however, due to the advantage of a low number of function evaluations and the extremely low global error, EFIRKT6 method is the best-performed method among the selected methods by generating the least maximum global error with similar computational time.

In addition, the TFIRKT6 method performs better than TFTDRKT6, ARK6 and TFRKT5 methods by obtaining the least maximum global error in integrating third-order ODEs with the oscillatory solution shown in Figures 6 to 8. The numerical curves are displayed in Figures 6 to 8 as the logarithms of maximum global errors are plotted against the computational time in seconds. Maximum global error obtained by all selected methods reduces readily as the step size becomes lower due to the convergence property acquired by all methods. Even though the complexity of the function for the TFIRKT6 method is higher compared to other methods, the number of function evaluations is one of the lowest in all selected methods, leading to low computation time. Improved Runge-Kutta-type methods collocate with the fitting technique have the largest advantage in accuracy compared to other methods due to the inclusion of a few previous steps in approximating the next term. In dealing with third-order application problems, the EFIRKT6 method is more proficient than the selected existing methods in solving thin film flow problems by generating the least maximum global error for all step sizes compared to existing methods.

CONCLUSION

In this article, we combined two-derivative Runge-Kutta-type methods with exponentially and trigonometrically-fitting techniques by developing exponentially-fitted and trigonometrically-fitted explicit improved two-derivative Runge-Kutta type methods with three-stage sixth-order denoted as EFIRKT6 and TFIRKT6 methods respectively. This article contributes to constructing improved two-derivative Runge-Kutta-type methods. It demonstrates how to adopt exponentially-fitting and trigonometrically-fitting techniques on the proposed methods to solve third-order periodic and exponential third-order ODEs with a much lower time of computation. The formulation comprises the previous step, b_{-i}, which vastly improves the accuracy of the existing two-derivative Runge-Kutta-type methods. Third and multiple fourth derivatives are formulated into the proposed methods to solve third-order ODEs in $u'''(t) = f(t, u(t), u'(t))$ with exponential or oscillatory solutions. The order conditions of generally improved two-derivative Runge-Kutta-type methods are proposed. Then exponential and trigonometrical techniques are implemented to construct frequency-reliant coefficients which integrate exactly suitable exponential and trigonometrical polynomials with exponential and periodic types.

Numerical tests prove the efficacy of EFIRKT6 and TFIRKT6 methods in solving third-order ODEs with exponential and trigonometrical solutions by generating the least maximum global error and low time of computation in similar step sizes when compared with other sixth-order existing methods. Through this research, a few topics can be explored.

EFIRKT6 and TFIRKT6 methods can be modified to solve and delay differential equations in the form of $u'''(t) = f(t, u(t), u(t-\tau)u'(t)u'(t-\tau))$ with the exponential and oscillatory solution. Also, the symmetric and symplectic properties can be adapted into exponentially-fitted and trigonometrically-fitted improved TDRKT methods to form modified methods with zero-dissipative and algebraically stable. Characterisations of symmetric and symplectic can be analysed, and numerical efficiency can be proved by solving oscillatory Hamiltonian systems effectively.

ACKNOWLEDGEMENT

The authors gratefully acknowledge the financial support from Universiti Putra Malaysia (UPM) under the Putra-IPB grant: GP-IPB/2017/9542402.

REFERENCE

Ahmad, N. A., Senu, N., & Ibrahim, Z. B., Othman, M., & Ismail, Z. (2020). High order three derivatie runge-kutta method with phase-fitting and amplification-fitting technique for periodic IVPs. *Malaysian Journal of Mathematical Sciences, 14*(3), 403-418.

Ahmad, N. A., Senu, N., & Ismail, F. (2019). Trigonometrically-fitted higher order two derivative runge-kutta method for solving orbital and related periodical IVPs. *Hacettepe Journal of Mathematics and Statistics, 48*(5), 1312-1323. https://doi.org/10.15672/HJMS.2018.568

Allogmany, R., & Ismail, F. (2020). Implicit three-point block numerical algorithm for solving third order initial value problem directly with applications. *Mathematics, 8*(10), Article 1771. https://doi.org/10.3390/math8101771

Al-Shimmary, A. F. A. (2017). Solving initial value problem using runge-kutta 6-th order method. *ARPN Journal of Engineering and Applied Sciences, 12*(13), 3953-3961.

Chen, Z., Li, J., Zhang, R., & You, X. (2015). Exponentially fitted two-derivative runge-kutta methods for simulation of oscillatory genetic regulatory systems. *Computational and Mathematical Methods in Medicine, 2015*, 1-14. https://doi.org/10.1155/2015/689137

D'Ambrosio, R., Paternoster, B., & Santomauro, G. (2014). Revised exponentially-fitted runge-kutta-nyström methods. *Applied Mathematics Letters, 30*, 56-60. https://doi.org/10.1016/j.aml.2013.10.013

Demba, M. A., Kumam, P., Watthayu, W., & Ismail, F. (2020). Embedded exponentially-fitted explicit runge-kutta-nyström method for solving periodic problems. *Computation, 8*(2), 165-178. http://www.ijpam.eu doi: 10.12732/ijpam.v111i2.3

Demba, M. A., Senu, N., & Ismail, F. (2016). A four-stage third-order symplectic explicit trigonometrically-fitted Runge-Kutta-Nyström method for the numerical integration of oscillatory initial-value problems. *International Journal of Pure and Applied Mathematics, 111*(2), 165-178.

Duffy, B. R., & Wilson, S. K. (1997). A third-order differential equation arising in thin-film flows and relevant to tanner's law. *Applied Mathematics Letters, 10*(3), 63-68. https://doi.org/10.1016/S0893-9659(97)00036-0

Franco, J. M., & Randez, L. (2018). An eighth-order exponentially fitted two-step hybrid method of explicit type for solving orbital and oscillatory problems. *Numerical Algorithms, 78*(1), 243-262. https://doi.org/10.1007/s11075-017-0374-1

Ghawadri, N., Senu, S., Ismail, F., & Ibrahim, Z. B. (2018). Exponentially fitted and trigonometrically fitted explicit modified runge-kutta type methods for solving $y''' = f(x, y, y')$. *Journal of Applied Mathematics, 2018*, Article 4029371. https://doi.org/10.1155/2018/4029371

Haweel, M. T, Zahran, O., & El-Samie, F. E. A. (2018). Adaptive polynomial method for solving third-order ODE with application in thin film flow. *IEEE Access*, *9*, 67874-67889.

Herrera, L. (2019). Causal heat conduction contravening the fading memory paradigm. *Entropy*, *21*(10), Article 950. https://doi.org/10.3390/e21100950

Hussain, K., Ismail, F., & Senu, N. (2017). Fourth-order improved runge-kutta method for directly solving special third-order ordinary differential equations. *Iranian Journal of Science and Technology, Transaction A: Science, 41*, 429-437. https://doi.org/ 10.1007/s40995-017-0258-1

Jikantoro, Y. D., Ismail, F., Senu, N., & Ibrahim, Z. B. (2018). A new integrator for special third order differential equations with application to thin film flow problem. *Indian Journal of Pure and Applied Mathematics*, *49*, 151-167. https://doi.org/10.1007/s13226-018-0259-6

Kumar, M., & Singh, N. (2012). Phase plane analysis and traveling wave solution of third order nonlinear singular problems arising in thin film evolution. *Computers & Mathematics with Application*, *64*(9), 2886-2895. https://doi.org/10.1016/j.camwa.2012.05.003

Lazer, A. C. (1966). The behavior of solutions of the differential equation $y''' + p(x)y' + q(x)y = 0$. *Pacific Journal of Mathematics*, *17*(3), 435-466.

Lee, K. C., Senu, N., Ahmadian, A., & Ibrahim, S. N. I. (2020). Numerical study of third-order ordinary differential equations using a new class of two derivative Runge-Kutta type methods. *Alexandria Engineering Journal, 59*(4), 2449-2467. https://doi.org/10.1016/j.aej.2020.03.008

Mei, L., Liu, C., & Wu, X. (2017). An essential extension of the finite-energy condition for extended runge-kutta-nyström integrators when applied to nonlinear wave equations. *Communications in Computational Physics, 22*(3), 742-764. https://doi.org/10.4208/cicp.OA-2016-0141

Monovasilis, T., & Kalogiratou, Z. (2021). High order two-derivative runge-kutta methods with optimized dispersion and dissipation error. *Mathematics*, *9*(3), Article 232. https://doi.org/10.3390/math9030232

Rabiei, F. (2011). Third-order improved runge-kutta method for solving ordinary differential equation. *International Journal of Applied Physics and Mathematics*, *1*(3), 191-194. http://doi.org/10.7763/IJAPM.2011.V1.37

Rabiei, F., & Ismail, F. (2012). Fifth-order improved runge-kutta method with reduced number of function evaluations. *Australian Journal of Basic and Applied Sciences*, *6*(3), 97-105.

Samat, F., & Ismail, E. S. (2020). Variable step exponentially fitted explicit sixth-order hybrid method with four stages for spring-mass and other oscillatory problems. *Symmetry*, *12*(3), Article 387. https://doi.org/10.3390/sym12030387

Senthilkumar, S., Lee, M., & Jeong, G. (2013). A modified improved runge-kutta fifth stage technique to study industrial robot arm. *International Journal of Pattern Recognition and Artificial Intelligence*, *27*(6), 1359004. https://doi.org/10.1142/S0218001413590040

Simos, T. E., & Williams, P. S. (1999). Exponential-fitted runge-kutta third algebraic order methods for the numerical solution of the schrödinger equation and related problems. *International Journal of Modern Physics C*, *10*(5), 839-851. https://doi.org/10.1142/S0129183199000656

Tang, X., & Xiao, A. (2020). Improved runge-kutta-chebyshev methods. *Mathematics and Computers in Simulation*, *174*, 59-75. https://doi.org/10.1016/j.matcom.2020.02.021

Zhai, H. Y., Zhai, W. J., & Chen, B. Z. (2018). A class of implicit symmetric symplectic and exponentially fitted Runge–Kutta–Nyström methods for solving oscillatory problems. *Journal of Inequalities and Applications*, *2018*, Article 321. https://doi.org/10.1186/s13660-018-1915-4

Zhang, Y., Che, H., Fang, Y., & You, X. (2013). A new trigonometrically fitted two-derivative runge-kutta method for the numerical solution of the schrödinger equation and related problems. *Journal of Applied Mathematics*, *2013*, Article 937858. http://dx.doi.org/10.1155/2013/937858